

## BMO( $\rho$ ) AND CARLESON MEASURES

BY

WAYNE STEWART SMITH

**ABSTRACT.** This paper concerns certain generalizations of BMO, the space of functions of bounded mean oscillation. Let  $\rho$  be a positive nondecreasing function on  $(0, \infty)$  with  $\rho(0+) = 0$ . A locally integrable function on  $\mathbf{R}^m$  is said to belong to  $\text{BMO}(\rho)$  if its mean oscillation over any cube  $Q$  is  $O(\rho(l(Q)))$ , where  $l(Q)$  is the edge length of  $Q$ .

Carleson measures are known to be closely related to BMO. Generalizations of these measures are shown to be similarly related to the spaces  $\text{BMO}(\rho)$ . For a cube  $Q$  in  $\mathbf{R}^m$ ,  $|Q|$  denotes its volume and  $R(Q)$  is the set  $\{(x, y) \in \mathbf{R}_+^{m+1} : x \in Q, 0 < y < l(Q)\}$ . A measure  $\mu$  on  $\mathbf{R}_+^{m+1}$  is called a  $\rho$ -Carleson measure if  $|\mu|(R(Q)) = O(\rho(l(Q))|Q|)$ , for all cubes  $Q$ .

L. Carleson proved that a compactly supported function in BMO can be represented as the sum of a bounded function and the balyage, or sweep, of some Carleson measure. A generalization of this theorem involving  $\text{BMO}(\rho)$  and  $\rho$ -Carleson measures is proved for a broad class of growth functions, and this is used to represent  $\text{BMO}(\rho)$  as a dual space. The proof of the theorem is based on a proof of J. Garnett and P. Jones of Carleson's theorem. Another characterization of  $\text{BMO}(\rho)$  using  $\rho$ -Carleson measures is a corollary. This result generalizes a characterization of BMO due to C. Fefferman. Finally, an atomic decomposition of the predual of  $\text{BMO}(\rho)$  is given.

**1. Introduction.** The space of functions of bounded mean oscillation, BMO, is closely related to certain measures known as Carleson measures. In this dissertation this relationship is extended to a generalization of BMO and a corresponding generalization of Carleson measures.

Let  $f$  be a real-valued locally integrable function on  $\mathbf{R}^m$  and  $Q$  a cube in  $\mathbf{R}^m$  with sides parallel to the coordinate axes. (We will throughout restrict ourselves to cubes with sides parallel to the coordinate axes.) We use the notation  $|Q|$  for the Lebesgue measure of  $Q$ ,  $l(Q)$  for the edge length of  $Q$ , and  $Q(f)$  for the average of  $f$  over  $Q$ , that is,

$$Q(f) = |Q|^{-1} \int_Q f(x) dx.$$

A growth function  $\rho$  is a positive nondecreasing function defined on  $(0, \infty)$  with  $\rho(0+) = 0$ . Let  $\rho$  be a growth function or the constant function 1. If

$$\sup_Q \rho(l(Q))^{-1} Q(|f - Q(f)|) = \|f\|_{\text{BMO}(\rho)} < \infty,$$

we say that  $f$  is in  $\text{BMO}(\rho)$ . Two growth functions are said to be equivalent if their quotient is bounded from above and from zero. Clearly if  $\rho$  and  $\eta$  are equivalent,  $\text{BMO}(\rho) = \text{BMO}(\eta)$ . It is well known that when we identify functions

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differing by a constant  $\text{BMO}(\rho)$  is a Banach space with the norm above. A good introduction to these spaces in the setting of the unit circle is in [12]. Dyadic  $\text{BMO}(\rho)$ ,  $\text{BMO}(\rho)_d$ , is the space of functions for which the above supremum taken over dyadic cubes only is finite. A dyadic cube is a cube of the form  $Q = \{x \in \mathbf{R}^m: a_j 2^{-n} < x_j < (a_j + 1)2^{-n}, 1 \leq j \leq m\}$ , where  $n$  and  $a_j$  are integers. It is clear that  $\text{BMO}(\rho) \subset \text{BMO}(\rho)_d$ , and simple examples show the inclusion is proper.  $\text{BMO}(\rho)_d$  is an easier space to work with than  $\text{BMO}(\rho)$  since two dyadic cubes either are disjoint or nested. However,  $\text{BMO}(\rho)$  is more interesting analytically since it is translation invariant, while  $\text{BMO}(\rho)_d$  is not. A useful technique in studying  $\text{BMO}(\rho)$  is to first prove a result on  $\text{BMO}(\rho)_d$  and then try to pass to the nondyadic case (see [6]).

When  $\rho$  is the constant function 1 we have  $\text{BMO}(\rho) = \text{BMO}$ . A certain type of measure on  $\mathbf{R}_+^{m+1} = \{(x, y): x \in \mathbf{R}^m, y > 0\}$  is closely related to BMO. Those positive measures  $\mu$  for which the map taking a function on  $\mathbf{R}^m$  to its Poisson integral on  $\mathbf{R}_+^{m+1}$  is bounded as a linear operator from  $L^p(dx)$  to  $L^p(d\mu)$ , for  $p > 1$ , have been characterized (see [7 and 14, 236]). The characterization, which is independent of  $p$ , is that

$$\sup_{Q \subset \mathbf{R}^m} |Q|^{-1} \mu(R(Q)) = M(\mu) < \infty.$$

Here  $R(Q) = \{(x, y) \in \mathbf{R}_+^{m+1}: x \in Q, y < l(Q)\}$ . This work was based on work of Carleson [1, 2] in dimension one on the unit circle, and these measures are now known as Carleson measures. The norm of the linear operator mentioned above is bounded by  $C_p M(\mu)^{1/p}$ . Here and throughout the paper  $C$  will denote a constant which may change with each usage, but which is independent of any variable in the equation in which it occurs. So here the constant is independent of the measure  $\mu$  and the function  $f \in L^p$ , but depends on  $p$  and the dimension.

C. Fefferman proved that a function  $f$  is in BMO if and only if

$$\int |f(x)|(1 + |x|^{m+1})^{-1} dx < \infty \quad \text{and} \quad \int y |\nabla f|^2 dx dy$$

is a Carleson measure [5, p. 145]. He also proved that BMO is the dual of the real variable version of the classical Hardy space  $H^1$ . In [3] Carleson proved constructively that a function  $f$  in BMO with compact support can be represented as the sum of a bounded function and the balyage, or sweep, of a Carleson measure with respect to any of a broad class of kernel functions, of which the Poisson kernel is representative. That is

$$f(x) = g(x) + \int_{\mathbf{R}_+^{m+1}} K_y(x - t) d\mu(t, y),$$

where  $g$  is bounded,  $\mu$  is a Carleson measure, and  $K(x)$  satisfies conditions (3.1) in §3. The converse is true, given slightly stronger conditions on  $K(x)$ . Carleson's theorem supplies another approach to the  $H^1$ -BMO duality.

In this paper analogous results are proven for  $\text{BMO}(\rho)$ , for a broad class of growth functions. It now is convenient to give some definitions pertaining to growth functions that we need, and to state some known results regarding them.

A growth function  $\rho$  is said to be regular if  $t \int_t^\infty (\rho(s)/s^2) ds = \tilde{\rho}(t) \leq C\rho(t)$ . The function  $\rho$  is of upper type  $\alpha$  if  $\rho(st) \leq Cs^\alpha \rho(t)$ , for all  $s \geq 1$  and  $t > 0$ . The

function  $\rho$  is of upper type less than  $\alpha$  if  $\rho$  is of upper type  $\beta$  for some  $\beta < \alpha$ . We use the notation  $\rho(\infty) = \lim_{t \rightarrow \infty} \rho(t) = \sup \rho(t)$ , so  $\rho(\infty) < \infty$  means that  $\rho$  is bounded. We state the results we need in the following lemma.

LEMMA 1.1. *Let  $\rho$  be a growth function.*

(i)  *$\tilde{\rho}$  is a growth function and  $\rho \leq \tilde{\rho}$ .*

(ii)  *$\rho$  is regular if and only if  $\rho$  is of upper type less than one.*

(iii) *If  $\rho$  is regular then  $\rho(t_1)/t_1 \leq C\rho(t_2)/t_2$ ,  $t_1 \geq t_2$ .*

For (i) see [12, p. 55] and for (ii) see [8, Lemma 4]. (iii) is a simple consequence of (ii) and the definitions. Note that a regular growth function  $\rho$  is equivalent to the continuous function  $\tilde{\rho}$ .

In §2 a theorem on  $\text{BMO}(\rho)_d$  is proved.

In §3 the  $\text{BMO}(\rho)$  version of Carleson's theorem on BMO is proved constructively, for  $\rho$  a regular growth function. The proof uses the theorem in §2, and is modeled on J. Garnett and P. Jones' proof of Carleson's theorem. A generalization of Carleson measures is introduced. The converse of the theorem is also given.

In §4 the results of §3 are used to represent  $\text{BMO}(\rho)$  as a dual space, for  $\rho$  regular and  $\rho(\infty) < \infty$ . As a corollary the  $\text{BMO}(\rho)$  version of Fefferman's characterization of BMO using Carleson measures is proved. An atomic decomposition of the predual of  $\text{BMO}(\rho)$  is also given. §4 is concluded with some remarks on transferring these results from  $\mathbf{R}^m$  to  $\mathbf{T}^m$ , the  $m$ -dimensional torus.

This paper is a slight revision of my dissertation, written under the supervision of Donald Sarason. I wish to express my appreciation for all the help and encouragement he gave me along the way.

## 2. A theorem about $\text{BMO}(\rho)_d$ .

THEOREM 2.1. *Let  $f \in \text{BMO}(\rho)_d$ , where  $\rho$  satisfies  $\rho(2t) \leq C\rho(t)$ , and let  $Q_0$  be a fixed dyadic cube. Then there exists a sequence  $\{Q_k\}$  of dyadic cubes,  $Q_k \subset Q_0$ , and a sequence  $\{a_k\}$  of real numbers such that*

$$(2.1) \quad \sum_{Q_k \subset Q} |a_k| |Q_k| \leq C \|f\| \rho(l(Q)) |Q|$$

for all dyadic cubes  $Q$ , and

$$(2.2) \quad f(x) - Q_0(f) = \sum a_k \chi_{Q_k}(x)$$

a.e. on  $Q_0$ . The constant  $C$  depends only on the dimension and  $\rho$ .

PROOF. To simplify things we can replace  $\rho$  by a growth function  $\bar{\rho}$  which satisfies

$$\bar{\rho}(t) \leq \rho(t) \leq 4\bar{\rho}(t)$$

(so  $\text{BMO}(\rho)_d = \text{BMO}(\bar{\rho})_d$ ) and so that if  $t_1 < t_2$ , then either

$$\bar{\rho}(t_1) = \bar{\rho}(t_2) \quad \text{or} \quad \bar{\rho}(t_1) \leq \frac{1}{4}\bar{\rho}(t_2).$$

We now go back to using  $\rho$  to denote this new function.

Assume without loss of generality that  $\|f\| = 1$ .

Set

$$G_1 = \{Q_k \subset Q_0: Q_k \text{ dyadic, } Q_k(|f - Q_0(f)|) > 2\rho(l(Q_k)), \text{ and } Q_k \text{ maximal}\}.$$

Maximality implies

$$(2.3) \quad Q_k(|f - Q_0(f)|) \leq M\rho(l(Q_k)).$$

To see this let  $Q_k^*$  be that dyadic cube with  $Q_k^* \supset Q_k$  and  $|Q_k^*| = 2^m |Q_k|$ . Since  $Q_k^* \notin G_1$  we have

$$Q_k^*(|f - Q_0(f)|) \leq 2\rho(l(Q_k^*)) \leq 2C\rho(l(Q_k))$$

and

$$Q_k(|f - Q_0(f)|) \leq \frac{|Q_k^*|}{|Q_k|} Q_k^*(|f - Q_0(f)|) = 2^m Q_k^*(|f - Q_0(f)|),$$

so (2.3) follows.

It is clear that the  $Q_k$  have disjoint interiors, by maximality, and  $\bigcup Q_k$  covers  $Q_0 \setminus \{f = Q_0(f)\}$  except for a set of measure zero, by the Lebesgue differentiation theorem.

We continue with our selection of cubes by induction. Suppose we have completed  $(n - 1)$  steps and have a family  $G_{n-1}$  of disjoint dyadic cubes. For each  $Q_j \in G_{n-1}$  set

$$G_1(Q_j) = \{Q_k \subset Q_j: Q_k \text{ dyadic, } Q_k(|f - Q_j(f)|) > 2\rho(l(Q_k)), \text{ and } Q_k \text{ maximal}\}$$

and  $G_n = \bigcup \{G_1(Q_j): Q_j \in G_{n-1}\}$ . Maximality again implies

$$(2.4) \quad Q_k(|f - Q_j(f)|) \leq M\rho(l(Q_k)).$$

Now, if  $Q_k \in G_n$  for some  $n$ , we can find  $Q_j \in G_{n-1}$  with  $Q_k \subset Q_j$ . Let

$$a_k = Q_k(f) - Q_j(f).$$

Note that (2.4) implies  $|a_k| \leq M\rho(l(Q_k))$ .

We claim  $\{\bigcup G_n\} = \{Q_k\}$  and  $\{a_k\}$  satisfy (2.1) and (2.2).

As a step toward proving (2.1) we prove that for  $Q_j \in G_{n-1}$

$$(2.5) \quad \sum_{\substack{Q_k \subset Q_j \\ Q_k \in G_n}} |a_k| |Q_k| \leq \frac{3}{4} M\rho(l(Q_j)) |Q_j|.$$

Write  $b_k = M\rho(l(Q_k))$ , so  $|a_k| \leq b_k$ . Then

$$\begin{aligned} \sum_{\substack{Q_k \subset Q_j \\ Q_k \in G_n}} |a_k| |Q_k| &\leq \sum b_k |Q_k| \\ &= \sum_{\rho(l(Q_k)) = \rho(l(Q_j))} b_k |Q_k| + \sum_{\rho(l(Q_k)) \leq \rho(l(Q_j))/4} b_k |Q_k| \\ &= \sum_1 + \sum_2 \end{aligned}$$

Clearly  $\sum_2 \leq \frac{1}{4} M\rho(l(Q_j)) |Q_j|$ , since the  $Q_k$  are disjoint. As for  $\sum_1$ ,

$$\begin{aligned} \sum_{\substack{\rho(l(Q_j)) = \rho(l(Q_k)) \\ Q_k \subset Q_j, Q_k \in G_n}} |Q_k| &< \sum \int_{Q_k} |f - Q_j(f)| \frac{1}{2\rho(l(Q_k))} \\ &\leq \int_{Q_j} |f - Q_j(f)| \frac{1}{2\rho(l(Q_j))} \leq \frac{1}{2} |Q_j| \end{aligned}$$

by the definition of  $G_1(Q_j)$ , the disjointness of the  $Q_k$ , and the equality  $\|f\| = 1$ . Since  $b_j \geq b_k$  we have  $\sum_1 \leq \frac{1}{2}b_j|Q_j|$ , and (2.5) is proved. Note that we have actually shown, for  $Q_j \in G_{n-1}$ ,

$$(2.5') \quad \sum_{\substack{Q_k \subset Q_j \\ Q_k \in G_n}} b_k|Q_k| \leq \frac{3}{4}b_j|Q_j|.$$

We are now in a position to prove (2.1). Fix a dyadic cube  $Q$  and set

$$G_1(Q) = \left\{ Q_j \in \bigcup G_n : Q_j \subset Q, Q_j \text{ maximal} \right\}$$

The  $Q_j$  in  $G_1(Q)$  are disjoint and

$$\sum_{Q_k \subset Q} |a_k| |Q_k| \leq \sum_{Q_k \subset Q} b_k |Q_k| = \sum_{Q_j \in G_1(Q)} \sum_{Q_k \subset Q_j} b_k |Q_k|$$

which, by induction and (2.5'), is

$$\leq \sum_{Q_j \in G_1(Q)} \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n b_j |Q_j| \leq C\rho(l(Q))|Q|,$$

since  $b_j \leq M\rho(l(Q))$  for all  $j$  and the  $Q_j$  are disjoint. (2.1) is now proved.

To prove (2.2), first let  $f_0(x) = Q_0(f)$ . We define  $f_n$  by induction. Assume  $f_{n-1}$  is defined. Let

$$f_n(x) = f_{n-1}(x) + \sum_{Q_k \in G_n} a_k \chi_{Q_k}(x).$$

Then, for  $Q_k \in G_n$ ,

$$f_n(x) = Q_k(f), \quad x \in Q_k.$$

Also, as before, for  $Q_j \in G_{n-1}$

$$\bigcup_{\substack{Q_k \subset Q_j \\ Q_k \in G_n}} Q_k \text{ covers } Q_j \setminus \{f = Q_j(f)\}$$

except for a set of measure zero, by the Lebesgue differentiation theorem and the definition of  $G_n$ .

This, the Lebesgue differentiation theorem again, and (2.4) now imply that  $f_n(x) \rightarrow f(x)$  a.e. on  $Q_0$ . Thus

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = Q_0(f) + \sum_{Q_k \in \bigcup G_n} a_k \chi_{Q_k}(x)$$

a.e. on  $Q_0$ , and so (2.2) holds and the proof is complete.

**3. Two theorems about BMO( $\rho$ ).** In this section a nondyadic version of Theorem 2.1 is stated and proved. Its proof, from Theorem 2.1, is modeled on the proof of Carleson's theorem on BMO [3] found by Garnett and Jones [6].

We need a generalization of Carleson measures.

DEFINITION. A measure  $\sigma$  on  $\mathbf{R}_+^{m+1}$  is called a  $\rho$ -Carleson measure if

$$|\sigma|(R(Q)) \leq C\rho(l(Q))|Q|$$

for all cubes  $Q$  in  $\mathbf{R}^m$ . The infimum of all constants  $C$  for which the above inequality holds is denoted by  $N(\sigma)$ .

Suppose  $K(x)$  is a differentiable function on  $\mathbf{R}^m$  satisfying

$$(3.1) \quad K(x) > 0, \quad K(x) \leq C(1 + |x|)^{-1-m},$$

$$\int K(x) dx = 1 \quad \text{and} \quad |\nabla K(x)| = o(1) \quad \text{as } |x| \rightarrow \infty.$$

For example, we can take  $K(x)$  to be the Poisson kernel on  $\mathbf{R}^m$ . We write  $K_y(x) = y^{-m}K(x/y)$ ,  $y > 0$ .

**THEOREM 3.1.** *Let  $\rho$  be regular and  $f \in \text{BMO}(\rho)$  have compact support. There is a  $\rho$ -Carleson measure  $\sigma$  such that*

$$(3.2) \quad f(x) = \int_{\mathbf{R}_+^{m+1}} K_y(x-t) d\sigma(t, y)$$

*a.e., and so that  $N(\sigma) \leq C\|f\|$ . The constant  $C$  depends only on  $K(x)$ ,  $m$  and  $\rho$ .*

**PROOF.** Suppose  $f$  has support  $S_0 = \{|x_i| \leq 1, 1 \leq i \leq m\}$ . For each  $\alpha \in S_0$  we have, by Theorem 2.1,

$$T_\alpha f(x) = f(x - \alpha) = \sum_{Q_k \subset Q_0} a_k^{(\alpha)} \chi_{Q_k}(x) + Q_0(f)$$

where  $Q_0 = \{|x_i| \leq 2, 1 \leq i \leq m\}$  and

$$(3.3) \quad \sum_{Q_k \subset Q} |a_k^{(\alpha)}| |Q_k| \leq C\|f\| \rho(l(Q)) |Q|$$

for all dyadic cubes  $Q$ . The above equation defines  $T_\alpha$ . Let

$$f_n^{(\alpha)}(x) = \sum_{l(Q_k)=2^{-n}} a_k^{(\alpha)} \chi_{Q_k}(x).$$

We can assume without loss of generality that  $Q_0(f) = 0$ , since if not we can add the cube  $Q_0$  to the family  $\{Q_k\}$  above, with  $a_0^{(\alpha)} = Q_0(f)$ . It is easy to check that  $|a_0^{(\alpha)}| \leq C\|f\| \rho(l(Q_0))$ , so (3.3) will still hold. We now have

$$T_\alpha f(x) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x).$$

So

$$\begin{aligned} f(x) &= \frac{1}{|S_0|} \int_{S_0} T_{-\alpha}(T_\alpha f(x)) d\alpha \\ &= \sum_{n=0}^{\infty} \frac{1}{|S_0|} \int_{S_0} f_n^{(\alpha)}(x + \alpha) d\alpha = \sum_{n=0}^{\infty} f_n(x), \end{aligned}$$

by the Lebesgue Dominated Convergence Theorem. For any cube  $Q$

$$\begin{aligned} \frac{1}{|Q|} \int_Q \sum_{2^{-n} \leq l(Q)} |f_n(x)| dx &\leq \sup_{\alpha \in S_0} \left\{ \frac{1}{|Q|} \int_Q \sum_{2^{-n} \leq l(Q)} |f_n^{(\alpha)}(x + \alpha)| dx \right\} \\ &\leq \sup_{\alpha \in S_0} \left\{ \frac{1}{|Q|} \sum_{Q_k \subset \tilde{Q}} |a_k^{(\alpha)}| |Q_k| \right\} \leq C\|f\| \rho(l(Q)), \end{aligned}$$

where  $Q$  and  $\tilde{Q}$  have the same center and  $l(\tilde{Q}) = 3l(Q)$ . Hence, for any  $\delta$  with  $0 < \delta < 1$ ,

$$d\sigma = \sum f_n(x) d\sigma_n,$$

where  $d\sigma_n$  is surface measure on  $\mathbf{R}^m \times \{y = \delta 2^{-n}\}$ , satisfies

$$\begin{aligned} |\sigma|(R(Q)) &\leq C\|f\|\delta^{-m}\rho(\delta^{-1}l(Q))|Q| \\ &\leq C\|f\|\delta^{-m-1}\rho(l(Q))|Q|, \end{aligned}$$

since  $\rho$  is upper type less than one by Lemma 1.1. That is,  $\sigma$  is a  $\rho$ -Carleson measure.

Let

$$\int K_y(x-t) d\sigma(t, y) = \sum_{n=0}^{\infty} f_n * K_{\delta 2^{-n}}(x) = \sum_{n=0}^{\infty} h_n(x).$$

We will show that, for sufficiently small  $\delta$ ,

$$(3.4) \quad \left\| \sum (f_n - h_n) \right\| \leq \frac{1}{2} \|f\|.$$

We will then do an iteration which will prove the theorem.

To prove (3.4) first note that for any cube  $Q$ , locally integrable function  $F$ , and constant  $a$  we have

$$\begin{aligned} \frac{1}{|Q|} \int_Q |F(x) - Q(f)| dx &\leq \frac{1}{|Q|} \int_Q |F(x) - a| dx + \left| \frac{1}{|Q|} \int_Q (F(x) - a) dx \right| \\ &\leq 2 \frac{1}{|Q|} \int_Q |F(x) - a| dx. \end{aligned}$$

Using this we see that for a fixed cube  $Q$  and a point  $x_0 \in Q$ ,

$$\begin{aligned} &\frac{1}{|Q|} \int_Q \left| \sum (f_n(x) - h_n(x) - Q(f_n - h_n)) \right| dx \\ &\leq 2 \sum_{2^{-n} \geq Al(Q)} \frac{1}{|Q|} \int_Q |(f_n - h_n)(x) - (f_n - h_n)(x_0)| dx \\ &\quad + 2 \sum_{l(Q) \leq 2^{-n} < Al(Q)} \frac{1}{|Q|} \int_Q |f_n(x) - h_n(x)| dx \\ &\quad + 2 \sum_{2^{-n} < l(Q)} \frac{1}{|Q|} \int_Q |f_n(x) - h_n(x)| dx \\ &= 2\sum_1 + 2\sum_2 + 2\sum_3 \end{aligned}$$

where  $A \geq 2$  is a constant we will fix later.

To estimate  $\sum_1$  we first need to establish

$$(3.5) \quad |f_n(x) - f_n(y)| \leq C2^n \|f\| \rho(2^{-n}) |x - y|.$$

To see this first note that we may assume that  $|x - y| < 2^{-n}$ , as otherwise (3.5) is an immediate consequence of the easily established inequality  $|f_n(x)| \leq C\|f\|\rho(2^{-n})$  for all  $x$ . This is true since it is true for each of the  $f_n^{(\alpha)}$ .

Now,

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq \frac{1}{|S_0|} \int_{S_0} \sum_{l(Q_k)=2^{-n}} |a_k^{(\alpha)}| |\chi_{Q_k}(x + \alpha) - \chi_{Q_k}(y + \alpha)| d\alpha \\ &\leq \frac{C\rho(2^{-n})\|f\|}{|S_0|} \int_{S_0} \sum_{l(Q_k)=2^{-n}} |\chi_{Q_k}(x + \alpha) - \chi_{Q_k}(y + \alpha)| d\alpha. \end{aligned}$$

The integrand is  $2\chi_E$ , where

$$E = \{\alpha \in S_0: x + \alpha \text{ and } y + \alpha \text{ fall in different } Q_k, l(Q_k) = 2^{-n}\},$$

and

$$\frac{1}{|S_0|} \int_{S_0} \chi_E \leq \sum_{i=1}^m \frac{|x_i - y_i|}{2^{-n}} \leq C|x - y|2^n.$$

(3.5) has now been proved.

Now,  $h_n = K_{\delta 2^{-n}} * f_n$  has the same continuity as  $f_n$  since  $\int K = 1$ . So

$$\begin{aligned} \sum_1 &\leq \sum_{2^{-n} \geq Al(Q)} \frac{1}{|Q|} \int_Q 2C2^n \|f\| \rho(2^{-n}) |x - x_0| dx \\ &\leq C\|f\|l(Q) \sum_{2^{-n} \geq Al(Q)} \frac{\rho(2^{-n})}{(2^{-n})^2} (2^{-n+1} - 2^{-n}) \\ &\leq C \frac{\|f\|}{A} Al(Q) \int_{Al(Q)}^1 \frac{\rho(t)}{t^2} dt \leq \frac{C\|f\|}{A} \rho(Al(Q)) \\ &\leq C\|f\| \frac{A^{1-r}}{A} \rho(l(Q)) \leq \frac{\|f\|\rho(l(Q))}{12} \end{aligned}$$

if  $A$  is large enough. The third to last inequality is just the definition of  $\rho$  being regular and the second to last inequality results from  $\rho$  being of upper type less than one, which is equivalent to  $\rho$  being regular by Lemma 1.1.

We have shown that  $2\sum_1 \leq \|f\|\rho(l(Q))/6$  for sufficiently large  $A$ .

To estimate  $\sum_2$  first note that by (3.5) and the estimate  $|f_n(x)| \leq C\|f\|\rho(2^{-n})$  we get that

$$|f_n(x) - f_n * K_{\delta 2^{-n}}(x)| \leq \varepsilon \|f\| \rho(2^{-n})$$

for all  $n$  if  $\delta$  is small enough, by standard reasoning. So,

$$\begin{aligned} 2\sum_2 &= 2 \sum_{l(Q) \leq 2^{-n} < Al(Q)} \frac{1}{|Q|} \int_Q |f_n(x) - f_n * K_{\delta 2^{-n}}(x)| dx \\ &\leq 2\varepsilon \|f\| \sum_{l(Q) \leq 2^{-n} < Al(Q)} \rho(2^{-n}) \leq C\varepsilon \|f\| \int_{l(Q)}^{Al(Q)} \frac{\rho(t)}{t} dt \\ &\leq C\varepsilon \|f\| Al(Q) \frac{\rho(l(Q))}{l(Q)} \leq \frac{\|f\|\rho(l(Q))}{6} \end{aligned}$$

if  $\varepsilon A$  is small, that is for sufficiently small  $\delta$ . The next to last inequality follows from Lemma 1.1(iii).



Now for  $\sum_3$ ,

$$\begin{aligned} 2\sum_3 &= 2 \frac{1}{|Q|} \int_Q \sum_{2^{-n} < l(Q)} |f_n(x) - f_n * K_{\delta 2^{-n}}(x)| dx \\ &\leq \sup_{\alpha} 2 \frac{1}{|Q|} \int_Q \sum_{2^{-n} < l(Q)} |f_n^{(\alpha)}(x) - f_n^{(\alpha)} * K_{\delta 2^{-n}}(x)| dx. \end{aligned}$$

Translation to the origin shows that it suffices to estimate the quantity following the ‘sup’ sign for  $\alpha = 0$ . Recall that  $\tilde{Q}$  is that cube with the same center as  $Q$  and  $l(\tilde{Q}) = 3l(Q)$ . Let  $Q^{(0)} = (\tilde{Q})^\sim$  and pave  $\mathbf{R}^m \setminus Q^{(0)}$  with cubes  $S^{(j)}$  congruent to  $Q$ . Let  $Q^{(j)} = \tilde{S}^{(j)}$ . So, by definition  $f_n^{(0)}$ ,

$$2\sum_3 \leq 2 \sum_j \sum_{Q_k \subset Q^{(j)}} |a_k^{(0)}| |Q_k| \frac{1}{|Q|} \int_Q \left| \frac{\chi_{Q_k}(x)}{|Q_k|} - \frac{\chi_{Q_k} * K_{\delta l(Q_k)}(x)}{|Q_k|} \right| dx,$$

as every dyadic cube  $Q_k$  must be in some  $Q^{(j)}$ , since  $2^{-n} < l(Q)$ .

Now note that

$$\int_{\mathbf{R}^m} \left| \frac{\chi_{Q_k}(x)}{|Q_k|} - \frac{\chi_{Q_k} * K_{\delta l(Q_k)}(x)}{|Q_k|} \right| dx$$

does not depend on  $l(Q_k)$ , as is seen by a change of scale. Thus, for  $\varepsilon > 0$ , we can choose a  $\delta$  so

$$\frac{1}{|Q|} \int_Q \left| \frac{\chi_{Q_k}(x)}{|Q_k|} - \frac{\chi_{Q_k} * K_{\delta l(Q_k)}(x)}{|Q_k|} \right| dx \leq \frac{\varepsilon}{|Q|}$$

for all  $k$ . Also, if  $Q_k \subset Q^{(j)}$ ,  $j \neq 0$ , then  $\chi_{Q_k}(x) = 0$  on  $Q$  and we get for the term to be estimated

$$\begin{aligned} \frac{1}{|Q|} \int_Q \frac{\chi_{Q_k} * K_{\delta l(Q_k)}(x)}{|Q_k|} dx &\leq \sup_{\substack{x \in Q \\ t \in Q_k}} K_{\delta l(Q_k)}(x - t) \\ &\leq C \delta l(Q_k) \frac{1}{\text{dist}(Q, Q_k)^{m+1}} \leq \frac{C \delta l(Q)}{\text{dist}(Q, Q^{(j)})^{m+1}} \end{aligned}$$

by (3.1) and since  $Q_k \subset Q^{(j)}$ . So

$$\begin{aligned} 2\sum_3 &\leq 2\varepsilon \sum_{Q_k \in Q^{(0)}} \frac{|a_k^{(0)}| |Q_k|}{|Q|} + C \delta l(Q) \sum_{j \neq 0} \frac{1}{\text{dist}(Q, Q^{(j)})^{m+1}} \sum_{Q_k \subset Q^{(j)}} |a_k^{(0)}| |Q_k| \\ &\leq C \varepsilon \|f\| \rho(2l(Q)) 9^m |Q| \frac{1}{|Q|} + C \delta l(Q) \sum_{j \neq 0} \|f\| \frac{\rho(l(Q^{(j)})) |Q^{(j)}|}{\text{dist}(Q, Q^{(j)})^{m+1}} \end{aligned}$$

by (3.3). And so

$$\begin{aligned} 2\sum_3 &\leq C \varepsilon \|f\| \rho(l(Q)) + C \delta \rho(l(Q)) \|f\| l(Q) \int_{\mathbf{R}^m \setminus Q^{(0)}} \frac{dx}{|x - x_0|^{m+1}} \\ &\leq C(\varepsilon + \delta) \|f\| \rho(l(Q)) \leq \frac{\|f\| \rho(l(Q))}{6} \end{aligned}$$

if  $\varepsilon$  and  $\delta$  are small, that is if  $\delta$  is sufficiently small.

Adding our estimates for  $2\sum_1$ ,  $2\sum_2$  and  $2\sum_3$  we get that

$$Q\left(\left|\sum(f_n - h_n - Q(f_n - h_n))\right|\right) \leq \|f\|\rho(l(Q))/2,$$

and (3.4) is valid for sufficiently small  $\delta$ .

We would be finished with the proof if we could iterate this. However,  $h_n$  does not have compact support, which is needed in the construction. But, conditions (3.1) on  $K$  imply that  $|\sum h_n(x)|$  and  $|\nabla \sum h_n(x)|$  tend to zero as  $|x|$  tends to infinity. This means that we can find a differentiable function,  $g$ , so that  $g$  has compact support and so

$$\left|\sum h_n(x) - g(x)\right| \quad \text{and} \quad \left|\nabla \left(\sum h_n(x) - g(x)\right)\right|$$

are both small. It is easy to see that this implies that  $\|\sum h_n - g\|$  is small. Using this we can do the required iteration.

Let  $0 < \lambda < 2^{-1}\|f\|$  and choose  $g$  as above so that  $\|g - \sum h_n\| < 2^{-1}\lambda$ . We write  $b_0$  for  $f$  and  $\mu_0$  for the measure  $\sigma$  we constructed above. We have

$$\begin{aligned} b_0(x) - \int K_y(x-t) d\mu_0(t, y) &= (f(x) - g(x)) + \left(g(x) - \sum h_n(x)\right) \\ &= b_1(x) + d_1(x), \end{aligned}$$

$b_1$  has compact support,  $\|b_1\| \leq \|f - \sum h_n\| + \|g - \sum h_n\| \leq \frac{3}{4}\|f\|$ , and  $\|d_1\| \leq 2^{-1}\lambda$ . We can now repeat the entire construction for  $b_1$ . Iterating this procedure while dividing  $\lambda$  by 2 at each step gives us, at step  $j$ ,

$$b_{j-1}(x) - \int K_y(x-t) d\mu_{j-1}(t, y) = b_j(x) + d_j(x),$$

where  $b_j$  has compact support,  $\|b_j\| \leq (\frac{3}{4})^j\|f\|$ ,  $\|d_j\| \leq 2^{-j}\lambda$ , and  $\mu_{j-1}$  is a  $\rho$ -Carleson measure with  $N(\mu_{j-1}) \leq C(\frac{3}{4})^{j-1}\|f\|$ . It follows that

$$(3.6) \quad f(x) - \int K_y(x-t) d\mu^n(t, y) = b_n(x) + \sum_{j=1}^n d_j(x),$$

where  $\mu^n = \sum_{j=0}^{n-1} \mu_j$  is a  $\rho$ -Carleson measure with  $N(\mu^n) \leq C\|f\| \sum_{j=0}^{n-1} (\frac{3}{4})^j \leq C\|f\|$ .

We need to examine each side of equation (3.6) as  $n$  tends to infinity. It is clear that  $b_n + \sum_{j=1}^n d_j$  is a Cauchy sequence in  $\text{BMO}(\rho)$ , and so it converges to a function,  $B$ , in  $\text{BMO}(\rho)$ . Clearly  $\|B\| \leq \lambda$ . We now consider the left side of equation (3.6). Let  $Q_i$  be the cube centered at the origin with  $l(Q_i) = 2^i$ . For each  $n$  we have  $|\mu^n|(R(Q_i)) \leq C\|f\|\rho(l(Q_i))|Q_i|$ . It follows that some subsequence of  $\{\mu^n\}$  converges in the weak-\* topology of measures on  $R(Q_i)$ . It is easy to show that the limit measure is a  $\rho$ -Carleson measure with  $\rho$ -Carleson constant at most  $C\|f\|$ . A standard diagonalization argument now produces a  $\rho$ -Carleson measure  $\mu$  on all of  $\mathbf{R}_+^{n+1}$ , with  $N(\mu) \leq C\|f\|$  and  $\mu$  the weak-\* limit of some subsequence of  $\{\mu^n\}$ . When we let  $n$  tend to infinity through the values of this subsequence, equation (3.6) becomes

$$f(x) - \int K_y(x-t) d\mu(t, y) = B(x),$$

where  $\|B\|$  can be made as small as we please. A second limiting argument like this one finishes the proof.

Notice that this proof does not require  $f$  to be compactly supported if we only require a representation of  $f$  on a fixed cube. We therefore have the following:

**COROLLARY 3.1.** *Let  $\rho$  be regular,  $f \in \text{BMO}(\rho)$ ,  $K(x)$  satisfy (3.1), and  $Q$  be a cube in  $\mathbf{R}^m$ . There is a  $\rho$ -Carleson measure  $\sigma$  such that*

$$f(x) - Q(f) = \int_{\mathbf{R}_+^{m+1}} K_y(x-t) d\sigma(t, y)$$

for almost all  $x \in Q$ , and so that  $N(\sigma) \leq C\|f\|$ .

The converse of Theorem 3.1 is valid if we have a better estimate of the rate of decay of  $\nabla K(x)$ . So now suppose that  $K(x)$  satisfies (3.1) and further that

$$(3.7) \quad |\nabla K(x)| \leq C(1 + |x|)^{-m-1}.$$

**THEOREM 3.2.** *Suppose that  $\sigma$  is a  $\rho$ -Carleson measure on  $\mathbf{R}_+^{m+1}$ , where  $\rho$  is regular. Further suppose that*

$$(3.8) \quad |\sigma|(\mathbf{R}^m \times (A, \infty)) < \infty$$

for sufficiently large  $A$ . Then

$$f(x) = \int_{\mathbf{R}_+^{m+1}} K_y(x-t) d\sigma(t, y)$$

is in  $\text{BMO}(\rho)$ , and  $\|f\| \leq CN(\sigma)$ .

Note that any  $\sigma$  coming from  $g \in \text{BMO}(\rho)$  via Theorem 3.1 satisfies (3.8).

**PROOF.** It is easy to see that conditions (3.1) and (3.8) imply that  $f(x)$  is locally integrable, and in particular it is defined almost everywhere.

Now let a cube  $Q$  be given. Let  $Q^*$  be the cube with the same center as  $Q$  and  $l(Q^*) = 2l(Q)$ . Let

$$f_1(x) = \int_{R(Q^*)} K_y(x-t) d\sigma(t, y) \quad \text{and} \quad f_2(x) = f(x) - f_1(x).$$

Then

$$\begin{aligned} \int_Q |f_1(x)| dx &\leq \int_{\mathbf{R}^m} K_y(x-t) dx \int_{R(Q^*)} d|\sigma|(t, y) \\ &\leq 1 \cdot 2^m |Q| \rho(2l(Q)) N(\sigma) \\ &\leq CN(\sigma) |Q| \rho(l(Q)). \end{aligned}$$

And so  $Q(|f_1 - Q(f_1)|) \leq CN(\sigma) \rho(l(Q))$ .

For  $f_2$  we first make the estimate for  $x_1, x_2 \in Q$  that

$$|f_2(x_1) - f_2(x_2)| \leq \int_{\mathbf{R}_+^{M+1} \setminus R(Q^*)} |K_y(x_1-t) - K_y(x_2-t)| d|\sigma|(t, y).$$

Now let  $Q^{(1)} = Q^*$ ,  $Q^{(n)} = (Q^{(n-1)})^*$  for  $n \geq 2$ . We now use (3.7) to estimate  $|K_y(x_1-t) - K_y(x_2-t)|$  when  $(t, y) \in R(Q^{(n)}) \setminus R(Q^{(n-1)})$ .

$$|K_y(x_1-t) - K_y(x_2-t)| = y^{-m} \left| K\left(\frac{x_1-t}{y}\right) - K\left(\frac{x_2-t}{y}\right) \right|,$$

and if  $t \in \mathbf{R}^m \setminus Q^{(n-1)}$  this is

$$\leq y^{-m} \left| \frac{x_1 - x_2}{y} \right| \frac{C}{(2^{n-2}l(Q)/y)^{m+1}} \leq \frac{C}{2^{(m+1)n}l(Q)^m}.$$

On the other hand, if  $y \geq 2^{n-1}l(Q)$ , then the above is

$$\leq y^{-m} \left| \frac{x_1 - x_2}{y} \right| C \leq \frac{Cl(Q)}{(2^n l(Q))^{m+1}} = \frac{C}{2^{(m+1)n}|Q|},$$

and so the same estimate holds here. Since for  $(t, y) \in R(Q^{(n)}) \setminus R(Q^{(n-1)})$  either  $t \in \mathbf{R}^m \setminus Q^{(n-1)}$  or  $y \geq 2^{n-1}l(Q)$ , the above estimate holds on  $R(Q^{(n)}) \setminus R(Q^{(n-1)})$ .

Now, going back to our estimate for  $|f_2(x_1) - f_2(x_2)|$ , we have

$$\begin{aligned} |f_2(x_1) - f_2(x_2)| &\leq \sum_{n=1}^{\infty} \int_{R(Q^{(n+1)}) \setminus R(Q^{(n)})} |K_y(x_1 - t) - K_y(x_2 - t)| d|\sigma|(t, y) \\ &\leq \frac{C}{|Q|} \sum_{n=1}^{\infty} \frac{1}{2^{(m+1)n}} \int_{R(Q^{(n+1)})} d|\sigma|(t, y) \\ &\leq \frac{C}{|Q|} \sum_{n=1}^{\infty} \frac{1}{2^{(m+1)n}} N(\sigma) 2^{m(n+1)} |Q| \rho(2^{n+1}l(Q)) \\ &\leq CN(\sigma)l(Q) \sum_{n=1}^{\infty} \frac{\rho(2^n l(Q))}{2^n l(Q)} \\ &\leq CN(\sigma)l(Q) \int_{l(Q)}^{\infty} \frac{\rho(s)}{s^2} ds \\ &\leq CN(\sigma)\rho(l(Q)) \end{aligned}$$

since  $\rho$  is regular.

Combining our estimates of the mean oscillation of  $f_1$  and  $f_2$  we get that

$$Q(|f - Q(f)|) \leq CN(\sigma)\rho(l(Q)),$$

and the proof is complete.

**4. Duality.** In this section we use Theorem 3.1 to represent  $BMO(\rho)$  as a dual space. S. Janson used other methods to do this [8]. We first give the definition of the space he showed to be the predual. Let  $\mathcal{S}$  be the Schwarz space of  $C^\infty$  rapidly decreasing functions on  $\mathbf{R}^m$ .  $H^1$  is the real variable version of the classical Hardy space  $H^1$  [5].

**DEFINITION.** Let  $\rho$  be a growth function and  $\phi \in \mathcal{S}$  satisfy  $\int \phi \neq 0$ . Then  $B(\rho, H^1, \phi)$  is the space of all distributions  $f$  satisfying

$$\|f\|_{B(\rho, H^1, \phi)} = \int_0^\infty \|\phi_t * f\|_{H^1} d\rho(t) < \infty.$$

The following results are in Janson's paper.

**LEMMA 4.1.** *Let  $\rho$  and  $\eta$  be equivalent growth functions and  $\phi, \psi \in \mathcal{S}$  with  $\int \phi, \int \psi \neq 0$ . Then*

- (i)  $B(\rho, H^1, \phi)$  is a Banach space;
- (ii)  $B(\rho, H^1, \phi) = B(\rho, H^1, \psi)$  with equivalent norms;
- (iii)  $B(\rho, H^1) = B(\eta, H^1)$  with equivalent norms.

The change in notation in (iii) is all right by (ii).

Janson showed that  $B(\rho, H^1)' = \text{BMO}(\rho)$ . The kernel function  $K(x)$  in Theorem 3.1 need not be in  $\mathcal{S}$ . This will allow us to represent the predual of  $\text{BMO}(\rho)$  as a space of harmonic functions, for example, which Janson's methods do not allow when we are working on  $\mathbf{R}^m$ . For the sake of simplicity and familiarity we will suppose that  $K(x)$  is the Poisson kernel for the definition and theorem below, although any kernel satisfying (3.1) and (3.7) could be used. The Poisson kernel is the function  $P(x) = c_m/(1 + |x|^2)^{(m+1)/2}$  (see [14, p. 61]).

DEFINITION. Let  $\rho$  be a growth function.  $C(\rho, H^1)$  is the space of all functions  $f(x, y)$  harmonic on  $\mathbf{R}_+^{m+1}$  satisfying

$$\|f\|_{C(\rho, H^1)} = \int_0^\infty \|f(x, y)\|_{H^1} d\rho(y) < \infty.$$

The  $H^1$  norm in the integrand is taken with respect to  $x$ .

The following lemma is proved in the same way as Lemma 4.1.

LEMMA 4.2. *Let  $\rho$  and  $\eta$  be equivalent growth functions. Then*

- (i)  $C(\rho, H^1)$  is a Banach space;
- (ii)  $C(\rho, H^1) = C(\eta, H^1)$  with equivalent norms.

We now need some results about  $H^1$ . For  $f(x, y)$  harmonic on  $\mathbf{R}_+^{m+1}$  we define  $f^+(x) = \sup\{|f(x, y)|: y > 0\}$ . Then  $\int f^+$  is one of several equivalent norms on  $H^1$  [5]. This is the norm we will use.  $H_0^1$  is the set of  $C^\infty$  rapidly decreasing functions in  $H^1$ .

LEMMA 4.3. *Let  $f \in H^1$  and  $f(x, y)$  be its harmonic extension to  $\mathbf{R}_+^{m+1}$ . Then*

- (i)  $\|f(\cdot, y_1)\|_{H^1} \leq \|f(\cdot, y_2)\|_{H^1}$  if  $y_2 \leq y_1$ ;
- (ii)  $f(\cdot, y) \rightarrow f$  in  $H^1$  as  $y \rightarrow 0$ ;
- (iii)  $H_0^1$  is dense in  $H^1$ .

(i) is immediate from our choice of norm. For (ii) and (iii) see [14, pp. 221, 225].

LEMMA 4.4. *Let  $f \in C(\rho, H^1)$ . Then  $f(\cdot, y) \rightarrow f$  in norm as  $y \rightarrow 0$ .*

PROOF. Let  $\varepsilon > 0$ . Choose  $N$  so large that

$$\int_0^{1/N} + \int_N^\infty \|f(x, y)\|_{H^1} d\rho(y) < \varepsilon.$$

Now choose  $\delta$  and  $\eta$  so that  $0 < \delta, \eta < 1/2N$  and  $\|f(x, \delta) - f(x, \eta)\|_{H^1} < \min(\varepsilon, \varepsilon/\rho(N))$ . This is possible by Lemma 4.3(ii). Now,

$$\begin{aligned} & \|f(x, \delta) - f(x, \eta)\|_{C(\rho, H^1)} \\ &= \int_0^{1/2N} + \int_{1/2N}^N + \int_N^\infty \|f(x, \delta + y) - f(x, \eta + y)\|_{H^1} d\rho(y) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By the choice of  $N, \delta$  and  $\eta$  we have  $I_1 + I_3 < 4\varepsilon$ . By Lemma 4.3(i),

$$I_2 < \|f(x, \delta) - f(x, \eta)\|_{H^1} \rho(N) < \varepsilon.$$

Thus  $\|f(x, \delta) - f(x, \eta)\|_{C(\rho, H^1)} < 5\varepsilon$  if  $\delta$  and  $\eta$  are sufficiently small. The lemma now follows by standard reasoning.

LEMMA 4.5.  $H_0^1$  is dense in  $C(\rho, H^1)$  if  $\rho(\infty) < \infty$ .

PROOF. By Lemma 4.4 we see that  $H^1$  is dense in  $C(\rho, H^1)$  and  $H_0^1$  is dense in  $H^1$  by Lemma 4.3. Now let  $\varepsilon > 0$  and  $f \in C(\rho, H^1)$ . Choose  $g \in H^1$  so that  $\|f - g\|_{C(\rho, H^1)} < \varepsilon$ . Now choose  $h \in H_0^1$  so that  $\|g - h\|_{H^1} < \varepsilon/\rho(\infty)$ . Then, by Lemma 4.3(i),

$$\begin{aligned} \|g - h\|_{C(\rho, H^1)} &= \int_0^\infty \|g(x, y) - h(x, y)\|_{H^1} d\rho(y) \\ &\leq \|g - h\|_{H^1} \rho(\infty) < \varepsilon. \end{aligned}$$

Thus  $\|f - h\|_{C(\rho, H^1)} < 2\varepsilon$ , and the lemma is proved.

We also need the following notion of a sequence adapted to a growth function, which can be found in [8].

DEFINITION. Let  $\rho$  be a regular growth function with  $\rho(\infty) < \infty$ . A sequence  $\{t_i\}_0^\infty$  of positive number is adapted to  $\rho$  if there are constants  $C_1$  and  $C_2$  so that

$$1 < C_1 \leq \rho(t_i)/\rho(t_{i+1}) \leq C_2, \quad i \geq 0,$$

and  $\rho(\infty) \leq C_2\rho(t_0)$ .

There is never any problem in choosing the  $t_i$  since the continuous function  $t \int_t^\infty (\rho(s)/s^2) ds$  is equivalent to  $\rho$ .

LEMMA 4.6. Let  $\rho$  be regular,  $\rho(\infty) < \infty$ ,  $\{t_i\}_0^\infty$  be adapted to  $\rho$  and  $g \in C(\rho, H^1)$ . Then

$$\sum_{i=0}^\infty \|g(x, t_i)\|_{H^1} \rho(2t_i)$$

is an equivalent norm on  $C(\rho, H^1)$ .

PROOF. The function  $\bar{\rho}(t) = \rho(t_i)$ ,  $t_{i+1} < t \leq t_i$ , is an equivalent growth function to  $\rho$  since  $\{t_i\}_0^\infty$  is adapted to  $\rho$ . By Lemma 4.2(ii)

$$\int_0^\infty \|g(x, y)\|_{H^1} d\bar{\rho}(y) = \sum_{i=0}^\infty \|g(x, t_i)\|_{H^1} (\rho(t_{i-1}) - \rho(t_i))$$

is an equivalent norm to  $\|g\|_{C(\rho, H^1)}$ . This implies the lemma since  $\{t_i\}_0^\infty$  is adapted to  $\rho$  and  $\rho(2t) \leq C\rho(t)$  for any regular growth function.

We need one last lemma, this time about BMO.

LEMMA 4.7. Let  $f \in \text{BMO}$  and  $Q_0$  be the cube centered at the origin with  $l(Q_0) = 1$ . Let  $Q_1$  be any cube centered at the origin with  $Q_0 \subset Q_1$ . Then

$$\int_{\mathbf{R}^m} \frac{|f(x) - Q_0(f)|}{1 + |x|^{m+1}} dx \leq C_0 \|f\| \quad \text{and} \quad \int_{\mathbf{R}^m} \frac{|f(x) - Q_1(f)|}{1 + |x|^{m+1}} dx \leq C_0 \|f\|.$$

PROOF. The inequality with  $Q_0$  is in [5]. The proof shows that if  $Q_1$  is used instead of  $Q_0$  a smaller upper bound results.

We now are ready to state and prove our duality results.

THEOREM 4.1. Suppose  $\rho$  is a regular growth function and  $\rho(\infty) < \infty$ . Then  $C(\rho, H^1)' = \text{BMO}(\rho)$  in the following sense:

(a) For  $f \in \text{BMO}(\rho)$ ,  $L(g) = \int fg$  is a bounded linear functional on  $C(\rho, H^1)$ , defined initially for  $g \in H_0^1$ , with  $\|L\| \leq C\|f\|_{\text{BMO}(\rho)}$ ;

(b) any continuous linear functional  $L$  on  $C(\rho, H^1)$  corresponds as in (a) to a unique function  $f$  in  $\text{BMO}(\rho)$ , with  $\|f\|_{\text{BMO}(\rho)} \leq C\|L\|$ .

PROOF. We will prove (a) first, so assume  $f \in \text{BMO}(\rho)$  and  $g \in H_0^1$ . This will suffice since  $H_0^1$  is dense in  $C(\rho, H^1)$  by Lemma 4.5. Since  $\text{BMO}(\rho) \subset \text{BMO}$ , we have by Lemma 4.7 that

$$\int \frac{|f(x) - Q_0(f)|}{1 + |x|^{m+1}} dx \leq C_0 \|f\|_{\text{BMO}} \leq C_0 \rho(\infty) \|f\|_{\text{BMO}(\rho)}.$$

This, the second part of Lemma 4.7 and the rapid decrease of  $g$  means that for  $\varepsilon > 0$  we can find a cube  $Q$  centered at the origin so that if  $F \in \text{BMO}(\rho)$  and  $\|F\| \leq \|f\|$ , then

$$(4.1) \quad \left| \int_{\mathbf{R}^m \setminus Q} (F - Q(F))g \right| \leq \sup_{x \notin Q} |g(x)(1 + |x|^{m+1})| \int_{\mathbf{R}^m} \frac{|F(x) - Q(F)|}{1 + |x|^{m+1}} dx < \varepsilon.$$

By Corollary 3.1 we have that

$$f(x) - Q(f) = \int_{\mathbf{R}_+^{m+1}} P_y(x - t) d\sigma(t, y)$$

for almost all  $x \in Q$ , where  $\sigma$  is a  $\rho$ -Carleson measure with  $N(\sigma) \leq C\|f\|$ . Let  $h(x) = \int_{\mathbf{R}_+^{m+1}} P_y(x - t) d\sigma(t, y)$ . Theorem 3.2 implies that  $\|h\| \leq C\|f\|$ .

Now

$$\int_{\mathbf{R}^m} fg = \int_{\mathbf{R}^m} (f - Q(f))g = \int_{\mathbf{R}^m} hg + \int_{\mathbf{R}^m \setminus Q} (f - Q(f))g - \int_{\mathbf{R}^m \setminus Q} hg,$$

and so we have that

$$\left| \int fg \right| \leq \left| \int hg \right| + (1 + c)\varepsilon$$

by (4.1), since  $Q(h) = 0$ . So we need to show

$$\left| \int hg \right| \leq C\|f\|_{\text{BMO}(\rho)} \|g\|_{C(\rho, H^1)}.$$

By Fubini's theorem

$$\int_{\mathbf{R}^m} hg = \int_{\mathbf{R}^m} \int_{\mathbf{R}_+^{m+1}} P_y(x - t) d\sigma(t, y) g(x) dx = \int_{\mathbf{R}_+^{m+1}} g(x, y) d\sigma(x, y).$$

At this point we claim that we can assume without loss of generality that for our function  $g$  in  $H_0^1$  there is a vector-valued function  $G$  so that  $G = (u_0, u_1, \dots, u_m)$ , where the functions  $u_j(x, y)$  satisfy the Cauchy-Riemann equations on  $\mathbf{R}_+^{m+1}$ ,  $G$  is continuous and rapidly decreasing on  $\overline{\mathbf{R}_+^{m+1}}$ ,  $|G| > 0$ ,  $\Delta|G| = O(1 + y + |x|)^{-m-\delta}$  for some  $\delta > 0$ , and  $u_0(x, y) = g(x, y)$ . We can make this assumption by a limiting argument found in [14, pp. 225–227]. Also see [5, p. 147]. It follows from this that

$$\int_{\mathbf{R}^m} \sup_{y > y_0} |G(x, y)| dx \leq C\|g(x, y_0)\|_{H^1}.$$

We also clearly have  $|g| \leq |G|$  on  $\mathbf{R}_+^{m+1}$ .

Now choose a sequence,  $\{t_i\}_0^\infty$ , adapted to  $\rho$ , with  $C_2 \leq 10$ . Let  $A_i = \{(x, y) \in \mathbf{R}_+^{m+1} : 2t_i \leq y < 2t_{i-1}\}$  for  $i \geq 0$ , where we take  $t_{-1}$  to be infinity. Let  $d\sigma_i(x, y) = \chi_{A_i}(x, y) d\sigma(x, y)$ . We then have

$$\left| \int_{\mathbf{R}_+^{m+1}} g(x, y) d\sigma(x, y) \right| \leq \sum_{i=0}^{\infty} \int_{\mathbf{R}_+^{m+1}} |G(x, y)| d|\sigma_i|(x, y).$$

We estimate these integrals separately. View  $|\sigma_i|$  as a measure on the half-space  $\mathbf{H}_{t_i} = \{(x, y) \in \mathbf{R}_+^{m+1} : t_i < y\}$ . The fact that  $\sigma$  is a  $\rho$ -Carleson measure implies that  $|\sigma_i|$  is a Carleson measure on  $\mathbf{H}_{t_i}$ , with Carleson constant at most

$$C\rho(2t_{i-1})N(\sigma) \leq CC_2\rho(2t_i)\|f\| \leq C\rho(2t_i)\|f\|.$$

At this point we would like to use the theorem on Carleson measures, mentioned in the Introduction, to estimate the above integrals. We cannot do this yet though, since we need an exponent greater than one on the integrands. We overcome this difficulty as in [5].

Let  $q = (m-1)/m$  if  $m > 1$ ,  $q = \frac{1}{2}$  if  $m = 1$ ,  $s(x) = |G(x, t_i)|^q$  and  $s(x, y)$  the Poisson integral of  $s$ . Then we have  $|G(x, t_i + y)| \leq s(x, y)^p$  for  $p = 1/q > 1$ ,  $s \in L^p(\mathbf{R}^m)$ , and

$$\|s\|_p^p = \int_{\mathbf{R}^m} |G(x, t_i)| dx \leq C\|g(x, t_i)\|_{H^1}.$$

For this see [14, pp. 222, 223].

We now have, by the Carleson measure theorem, that

$$\begin{aligned} \int_{\mathbf{R}_+^{m+1}} |G(x, y)| d|\sigma_i|(x, y) &= \int_{\mathbf{H}_{t_i}} |G(x, y)| d|\sigma_i|(x, y) \\ &\leq \int_{\mathbf{H}_{t_i}} s(x, y - t_i)^p d|\sigma_i|(x, y) \\ &\leq C\rho(2t_i)\|f\| \|s\|_p^p \leq C\rho(2t_i)\|f\| \|g(x, t_i)\|_{H^1}. \end{aligned}$$

Putting the above estimates together we have

$$\begin{aligned} \left| \int hg \right| &\leq C\|f\|_{\text{BMO}(\rho)} \sum_{i=0}^{\infty} \|g(x, t_i)\|_{H^1} \rho(2t_i) \\ &\leq C\|f\|_{\text{BMO}(\rho)} \|g\|_{C(\rho, H^1)}, \end{aligned}$$

where the last inequality is due to Lemma 4.6.

The proof of (a) is now complete.

We now go on to the proof of (b). Suppose  $L$  is a bounded linear functional on  $C(\rho, H^1)$ . As noted before, we only need to find a representation of  $L$  on  $H_0^1$  since this is a dense subspace. If  $g \in H_0^1$ , then  $\|g\|_{C(\rho, H^1)} \leq \rho(\infty)\|g\|_{H^1}$ . It follows that  $L$  induces a bounded linear functional on  $H^1$ . Since  $(H^1)' = \text{BMO}$ , we have  $L(g) = \int fg$  for some  $f$  in  $\text{BMO}$ . We will show that  $f$  actually is in  $\text{BMO}(\rho)$ .

Let a cube  $Q$  in  $\mathbf{R}^m$  be fixed and let  $g$  be any function such that  $\text{support}(g) \subset Q$ ,  $|g(x)| \leq |Q|^{-1}$  and  $\int g = 0$ . We will estimate  $\|g\|_{C(\rho, H^1)}$ . We have

$$(4.2) \quad g(x, y) = \int_{\mathbf{R}^m} P_y(x - t)g(t) dt = \int_Q P_y(x - t)g(t) dt.$$



By Taylor's theorem we have

$$(4.3) \quad P_y(x-t) = C_1 + C_2 t$$

for each  $t \in Q$  and  $(x, y) \in \mathbf{R}_+^{m+1}$ , and

$$(4.4) \quad \begin{aligned} |C_2| &\leq \sup_{t \in Q} \frac{1}{y^{m+1}} \left| \nabla P \left( \frac{x-t}{y} \right) \right| \\ &\leq \sup_{t \in Q} C(y + |x-t|)^{-1-m} \end{aligned}$$

by the estimate (3.7) for  $K(x) = P(x)$ . Notice that  $C_1$  will not contribute to the above integral since  $\int g = 0$ .

Now let  $Q^{(0)} = Q$  and let  $Q^{(n)}$  have the same center as  $Q^{(n-1)}$  with  $l(Q^{(n)}) = 2l(Q^{(n-1)})$ , for  $n \geq 1$ . Then

$$\begin{aligned} \|g\|_{C(\rho, H^1)} &= \int_{\mathbf{R}_+^{m+1}} \sup_{s > y} |g(x, s)| dx d\rho(y) \\ &= \sum_{n=0}^{\infty} \int_{R(Q^{(n)}) \setminus R(Q^{(n-1)})} \sup_{s > y} |g(x, s)| dx d\rho(y) \\ &= \sum_{n=0}^{\infty} I_n, \end{aligned}$$

where we take  $Q^{(-1)} = \emptyset$ . Since  $|g(x, s)| \leq |Q|^{-1}$  everywhere, we have

$$I_0 + I_1 \leq |Q|^{-1} 2^m |Q| \rho(2l(Q)) \leq C \rho(l(Q)).$$

Now, using (4.2), (4.3) and (4.4), by the same estimates used in the proof of Theorem 3.2 we get that

$$I_n \leq \frac{Cl(Q)}{(l(Q^{(n)}))^{m+1}} |Q^{(n)}| \rho(l(Q^{(n)})) = \frac{C \rho(2^n l(Q))}{2^n}, \quad \text{for } n \geq 2.$$

As before we get  $\sum_{n=2}^{\infty} I_n \leq C \rho(4l(Q)) \leq C \rho(l(Q))$ , and so  $\|g\|_{C(\rho, H^1)} = \sum_{n=0}^{\infty} I_n \leq C \rho(l(Q))$ .

Thus, for any such  $g$ ,

$$\left| \int f g \right| \leq C \|L\| \rho(l(Q)).$$

So, for  $g(x) = |Q|^{-1} \text{sign}(f(x) - Q(f)) \chi_Q(x)$ , we have

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f - Q(f)| &= \int_Q g(f - Q(f)) = \int_Q (g - Q(g)) f \\ &\leq 2C \|L\| \rho(l(Q)). \end{aligned}$$

Therefore  $f \in \text{BMO}(\rho)$  with  $f \leq C \|L\|$ . The proof of (b) is complete except for showing that  $f$  is unique. But this is easy since the above argument shows that if  $f_1$  and  $f_2$  correspond to  $L$ , then  $Q(|f_1 - f_2 - Q(f_1 - f_2)|) = 0$  for all cubes  $Q$ . This means  $\|f_1 - f_2\|_{\text{BMO}(\rho)} = 0$ .

Theorem 4.1 has the following corollary, which is analogous to a characterization of BMO found by Fefferman [5, p. 145]. That (a) implies (b) seems to be well known, at least for dimension one on the unit circle [12, p. 61]. The same proof

works on  $\mathbf{R}^m$ , and so will be omitted here. That (b) implies (a) answers a question by Sarason, asked in the setting of the unit circle. C. Mueller [11] and myself (unpublished) have found proofs in this setting. S. Janson [9] has found another proof that works on  $\mathbf{R}^m$ . The proof below is similar to the proof of the corresponding result for BMO. In particular we need to have the predual of  $\text{BMO}(\rho)$  represented as a space of harmonic functions. We write  $f(x, y)$  and  $g(x, y)$  for the Poisson integrals of the functions  $f$  and  $g$  on  $\mathbf{R}^m$ , and  $|\nabla f|^2$  for  $|\partial f/\partial y|^2 + \sum_{j=1}^m |\partial f/\partial x_j|^2$ .

**COROLLARY 4.1.** *Let  $\rho$  be a regular growth function with  $p(\infty) < \infty$ . Then the following are equivalent:*

(a)  $f \in \text{BMO}(\rho)$ .

(b)

$$\int_{\mathbf{R}^m} \frac{|f(x)|}{1 + |x|^{m+1}} dx < \infty \quad \text{and} \quad y |\nabla f(x, y)|^2 dx dy$$

is a  $\rho^2$ -Carleson measure on  $\mathbf{R}_+^{m+1}$ .

**PROOF.** We only prove that (b) implies (a). The first condition in (b) implies that the Poisson integral of  $f$  exists, and so the second condition makes sense. In view of Theorem 4.1 it suffices to show that any  $f$  satisfying condition (b) induces a bounded linear functional on  $C(\rho, H^1)$ , given by  $L(g) = \int fg$  for  $g$  in  $H_0^1$ . Fefferman's theorem on BMO shows that (b) implies  $f$  is in BMO. In the proof of that theorem it is shown that for  $f$  in BMO and  $g$  in  $H_0^1$

$$\int_{\mathbf{R}^m} fg = 2 \int_{\mathbf{R}_+^{m+1}} y \nabla f \cdot \nabla g dx dy,$$

and so it will suffice to estimate the integral on the right.

We again choose a sequence adapted to  $\rho, \{t_i\}_0^\infty$ , with  $C_2 \leq 10$ , and as before we let  $A_i = \{(x, y) \in \mathbf{R}_+^{m+1} : 2t_i \leq y < 2t_{i-1}\}$ ,  $i \geq 0$ , where we take  $t_{-1}$  to be infinity. We will estimate the above integral over each  $A_i$  separately.

We now make the same assumptions on  $g$  as were made in the proof of Theorem 4.1, so we have a vector-valued function  $G$  satisfying the conditions listed there.

Now,

$$\begin{aligned} \left| \int_{A_i} y \nabla f \cdot \nabla g dx dy \right| &\leq \int_{A_i} y |\nabla f| |\nabla G| dx dy \\ &\leq \left( \int_{A_i} y |\nabla f|^2 |G| dx dy \right)^{1/2} \left( \int_{A_i} y |\nabla G|^2 |G|^{-1} dx dy \right)^{1/2} \\ &= (I_1 I_2)^{1/2}. \end{aligned}$$

The same argument used in the proof of Theorem 4.1 shows that

$$I_1 \leq C \rho^2(2t_i) \|g(x, t_i)\|_{H^1}.$$

To estimate  $I_2$  we use the inequality  $|\nabla G|^2 |G|^{-1} \leq (m+1)\Delta|G|$  [14, p. 217]. So

$$\begin{aligned} (m+1)^{-1} I_2 &\leq C \int_{A_i} y \Delta |G| dx dy \\ &\leq \int_{\mathbf{H}_{t_i}} (y+t_i) \Delta |G|(x, y+t_i) dx dy \\ &\leq 2 \int_{\mathbf{R}_+^{m+1}} y \Delta |G|(x, y+t_i) dx dy \\ &= 2 \int_{\mathbf{R}^m} |G(x, t_i)| dx \leq C \|g(x, t_i)\|_{H^1}. \end{aligned}$$

The equality follows from Green's theorem [14, p. 87].

Putting the above estimates together we see that

$$\left| \int_{A_i} y \nabla f \cdot \nabla g dx dy \right| \leq C \rho(2t_i) \|g(x, t_i)\|_{H^1}.$$

Lemma 4.6 now shows that

$$\left| \int_{\mathbf{R}_+^{m+1}} y \nabla f \cdot \nabla g dx dy \right| \leq C \|g\|_{C(\rho, H^1)}.$$

This finishes the proof.

By keeping track of the various constants in this proof one can show that if  $\sigma(f)$  is the  $\rho^2$ -Carleson measure in condition (b), then  $N(\sigma(f))$  is comparable to  $\|f\|^2$ .

As another corollary to Theorem 4.1 we give an atomic decomposition of  $C(\rho, H^1)$ , using the same type of reasoning used in [4] to give the atomic decomposition of the  $H^p$  spaces.

DEFINITION. A  $\rho-1$ -atom is a function  $a(x)$  on  $\mathbf{R}^m$  satisfying:

- (i) The support of  $a(x)$  is contained in a cube  $Q$ ;
- (ii)  $|a(x)| \leq (\rho(l(Q))|Q|)^{-1}$ ;
- (iii)  $\int a = 0$ .

COROLLARY 4.2. A distribution  $f$  is in  $C(\rho, H^1)$  if and only if it can be represented in the form

$$(4.5) \quad \sum_{i=0}^{\infty} \lambda_i a_i(x)$$

for  $a_i$   $\rho-1$ -atoms and  $\lambda_i$  real numbers satisfying  $\sum_{i=0}^{\infty} |\lambda_i| < \infty$ . Also

$$|||f||| = \inf \left\{ \sum_{i=0}^{\infty} |\lambda_i| : (4.5) \text{ represents } f \right\}$$

is an equivalent norm on  $C(\rho, H^1)$ .

PROOF. Let  $\mathcal{C}$  be the space of all distributions that can be represented in the form (4.5). It is clear that  $|||\cdot|||$  is a norm on  $\mathcal{C}$ . The proof of part (b) of Theorem 4.1 shows that any  $\rho-1$ -atom is in  $C(\rho, H^1)$ , with its norm at most one. The series (4.5) therefore converges in  $C(\rho, H^1)$ , since  $\sum_{i=0}^{\infty} |\lambda_i| < \infty$ . It is easy to see that  $\mathcal{C}$  is complete. It is clear that a function  $f$  in  $\text{BMO}(\rho)$  defines a bounded linear

functional on  $\mathcal{C}$ , given by  $L(a) = \int fa$  for an atom  $a$ . Also, any such functional must be so represented, by the argument used to prove part (b) of Theorem 4.1. Since  $\mathcal{C}$  and  $C(\rho, H^1)$  have the same dual space, represented in the same way, and  $\mathcal{C} \subset C(\rho, H^1)$ , the Hahn-Banach theorem implies  $\mathcal{C} = C(\rho, H^1)$ , with equivalent norms. The proof is complete.

It should be remarked that if we consider functions on  $\mathbf{T}^m$ , the  $m$ -dimensional torus, all the results in this paper are still valid. The proofs must be rewritten with the torus in mind, and there are simplifications since compactness is no problem. In Corollary 4.2 we would have to include the atom  $a_0(x) \equiv \rho(2\pi)^{-1}(2\pi)^{-m}$ , since  $C(\rho, H^1)$  then contains nonzero constant functions.

$\rho$ -Carleson measures on the unit disk of the complex plane have been studied for  $\rho(t) = t^\alpha$ . See, for example, [10 and 13]. The methods of this paper generalize as well as simplify some of these results.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII AT MANOA, HONOLULU, HAWAII 96822